Mean-field theory of hot sandpiles

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We introduce a temperaturelike parameter T in a mean-field theory of sandpiles. We show that, in contrast with evolution and growth models, self-organized criticality in sandpiles exists at all $T<\infty$ with exponents that do not depend on the temperature. We provide an explanation for the difference in the behavior of sandpiles and the other models. [S1063-651X(97)00905-7]

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The idea of self-organized criticality (SOC) was introduced by Bak, Tang, and Wiesenfeld [1] to explain the ubiquity of spacial and temporal fractals in nature. In their pioneering work [1], they proposed sandpiles as the simplest model of SOC. In this paper we study the role of temperaturelike parameter T in a mean-field theory of sandpiles. We consider the model considered to be similar to the two-state sandpile model, studied numerically by Manna [2].

It was previously shown [3] that temperature plays an important role in other models that exhibit SOC behavior, such as Sneppen growth models [4] (where the introduction of a nonzero temperature changes the model universality class) or the Bak-Sneppen evolution model [5] (where SOC is possible only at zero temperature). On the other hand, in recent work [6] Caldarelli, Maritan, and Vendruscolo introduced a temperaturelike parameter in an ordinary sandpile model, studied it numerically, and showed that SOC exists at all temperatures, although with exponents, which change continuously with temperature. In this paper we introduce a temperaturelike parameter, in a spirit similar to that of the work [6], in the mean-field theory of sandpiles proposed by us earlier [7]. We show that in our model SOC exists at all finite temperatures and that exponents in our mean-field model do not depend on the temperature and coincide with the usual mean-field exponents [8–10].

We consider the following variant of the usual sandpile model [7]: on each of N sites we define an integer number z_i that represents the number of sand grains at this site. At every time step, each z_i is increased by 1 with a probability $h \ll 1$: $z_i \rightarrow z_i+1$. If z_i exceeds $z_c=1$, then at the next time step it topples $z_i \rightarrow z_i-2$ and releases two grains of sand. With a probability $\epsilon \ll 1$, one of these grains leaves the system and the other one lands on a randomly chosen site, and the height of this site is increased by 1. With a probability $1-\epsilon$, both grains land somewhere in the system. To emulate the effects of the temperature, we allow a site with 1 grain of sand to topple with probability $\pi=\exp(-1/T)$, if that site received a grain of sand at the preceding time step. After the toppling, with a probability $1-\epsilon$, a grain is transferred to a randomly chosen site, whereas with a probability ϵ this grain is lost. $\epsilon \ll 1$ is a measure of the dissipation and represents the probability of a site to be on the boundary of the system. The SOC state is reached by first letting $h \rightarrow 0$, and then $\epsilon \rightarrow 0$.

The dynamics of the avalanches are controlled by m_j, the average number of grains created by one grain at the next time step. By mapping this model to a branching process it is straightforward to show [7,11] that if $m-1$, then the system is in a critical state and the distributions of both avalanche sizes $D_3(s)$ and durations $D_t(t)$ are described by power laws $D_s(s) \sim s^{-\tau+1}$ and $D_t(t) \sim t^{-b}$, with the usual values for the exponents $\tau=5/2$ and $b=2$. If $m \neq 1$, then the criticality is destroyed and there appears a characteristic avalanche size s_{co} and characteristic avalanche duration t_{c0}: $D_s(s) \sim s^{-\tau-1} \exp(-s/s_{co})$ and $D_t(t) \sim t^{-b} \exp(-t/t_{co})$ with $s_{co} \sim 1/(1-m)^2$ and $t_{co} \sim 1/(1-m)$.

To find the value of m in the steady state, eventually reached by the system, one can write the master equation for this model, which is, as usual, a balance equation, stating that the change in the number of sites of a given height i equals the number of sites that change their height as to i minus the number of sites of height i, which change their heights from i to some other value:

$$P_i(t+1)=P_i(t)+\sum_{j=0}^{\infty} [P_j(t)T_{ji}(t)-P_i(t)T_{ij}(t)], \quad (1)$$

where $P_i(t)$ is the fraction of sites of height i at time t and T_{ij} is the transition probability for any given site of height i to change its height to j. If one makes an approximation that there are only sites with heights of 0, 1, 2, and 3 and that $P_2 \ll 1$ and $P_3 \ll 1$, then the relevant T_{ij} are

$$T_{01}=T_{21}=T_{32}=hv_0+(1-h)v_1,$$

$$T_{02}=hv_1+(1-h)v_2,$$

$$T_{03}=T_{23}=hv_2,$$

$$T_{20}=T_{31}=(1-h)v_0,$$

$$T_{10}=\pi A_c T_{20},$$

$$T_{12}=(1-\pi A_c)T_{01}+\pi A_c T_{02},$$

$$T_{13}=(1-\pi A_c)T_{02}+\pi A_c T_{03},$$

$$T_{30}=0,$$

where

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$$v_0 = 1 - A + A^2/2,$$

$$v_1 = A - A^2,$$

$$v_2 = A^2/2.$$

$A = (2-\epsilon)[P_2(t) + P_3(t)] + (1-\epsilon)\pi A_c P_i(t)$ is the number of grains of sand, redistributed at time step t, and $A_c = T_{01}[P_0(t-1) + P_2(t-1)]/P_1(t)$ is the probability of a site with 1 grain to have received a grain during the previous time step. A_c is the ratio of the total number of sites with 1 grain to the number of sites that 0 or 2 grains at the preceding time step and received 1 grain, thus turning into a site with 1 grain. The transition probabilities T_{ij} are calculated by a straightforward generalization of the procedure described in [7].

In this approximation one finds in the usual limit $\epsilon, h/\epsilon \to 0$ that in the steady state $P_0 = 1/(2-\pi)$, $P_1 = (1-\pi)/(2-\pi)$, $P_2, P_3 \to 0$. In this limit, $m = \pi P_0 + 2P_1 = 1$ and the system self-organizes to a critical steady state characterized by the absence of a characteristic time or length scale, independent of the temperature, as long as it is finite. The case $T = \infty$ ($\pi = 1$) is special because here not only is the average number of grains created by one grain, m, 1, but the actual number of grains created by one grain at every time step is exactly 1. If $T = \infty$, then in the steady state there is only one sand grain in the system and that grain is hopping around forever, representing one infinite avalanche.

To summarize, we have introduced a temperaturelike parameter in the mean-field model of sandpiles and showed that in our model SOC exists at all $T < \infty$, in contrast with other models [3] and in agreement with the findings of [6]. Thus, our results demonstrate that SOC in sandpile models is robust with respect to thermal fluctuations, which can help explain why fractals in nature are not destroyed by these fluctuations. The difference between effects of the temperaturelike parameter on the SOC models considered in [3] and sandpiles is due to the fact that the dynamics of the models considered in [3] are extremal. In these extremal models, at every time step the dynamics are restricted to the ''smallest'' site, the site where the pinning force is minimal (growth models) or the site representing the species with the smallest barrier to mutate (evolution model). In such models, the introduction of even an infinitesimally small temperature allows all sites to participate in the dynamics. Because of that, the chances of choosing the smallest site become negligible; although other sites have a smaller probability of being chosen, there are a lot of them. As a result, the dynamics of the model change completely. In contrast, in sandpiles the introduction of a temperaturelike parameter is equivalent to changing local rules, which do not affect the global dynamics. Thus sandpiles are truly self-organized and critical, unlike their cousins involving extremal dynamics.

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[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).
[2] S. S. Manna, J. Phys. A **24**, L363 (1991).
[3] M. Vergeles, Phys. Rev. Lett. **75**, 1969 (1995).
[4] K. Sneppen, Phys. Rev. Lett. **69**, 3539 (1992).
[5] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
[6] G. Caldarelli, A. Maritan, and M. Vendruscolo, Europhys. Lett. **35**, 481 (1996).
[7] M. Vergeles, A. Maritan, and J. R. Banavar, Phys. Rev. E **55**, 1998 (1997); see also K. B. Lauritsen, S. Zapperi, and H. E. Stanley, *ibid.* **54**, 2483 (1996).
[8] C. Tang and P. Bak, J. Stat. Phys. **51**, 797 (1988).
[9] S. Zapperi, K. B. Lauritsen, and H. E. Stanley, Phys. Rev. Lett. **75**, 4071 (1995).
[10] S. A. Janowsky and C. A. Laberge, J. Phys. A **26**, L973 (1993).
[11] W. Feller, *An Introduction to Probability Theory and Its Applications*, 2nd ed. (Wiley, New York, 1971), Vol. 1.